Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Certain Surfaces with Plane or Spherical Lines of Curvature.

By L. P. Eisenhart.

§1. Introduction.

Elsewhere we have considered the surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces; we have called them the *A-surfaces.*[*] We have found that they admit of transformations into new A-surfaces in a way which is a generalization of the Bäcklund transformations of pseudospherical surfaces. In the present paper we wish to study the A-surfaces with plane or spherical lines of curvature in one system and to apply the earlier results to these particular surfaces.

After a brief recall in §2 of the needed results of our former study, we proceed in §3 to the discussion of A-surfaces with plane lines of curvature in one or both systems. We show that the moulure surfaces are the only A-surfaces with the lines in both systems plane, and then find the character of the curves on the sphere which represent plane lines in one system on A-surfaces.

In §4 we consider the A-surfaces of the Joachimstal type, first in terms of the parameters used in §§2 and 3, and then in terms of a set of parameters so chosen as to enable us to give the cartesian coordinates of the surface in a simple form. We find that among all the A-surfaces with the same spherical representation and having plane lines of curvature in one system there is an infinity of the Joachimstal type, the pseudospherical surface being one of the latter.

In §5 after noting that for all A-surfaces with plane lines of curvature in one system the planes of the lines envelope a cylinder, which reduces to a line for the surfaces of Joachimstal, we make use of an investigation by Raffy, *Sur les surfaces a lignes de courbure planes dont les plans enveloppent un cylindre* (*Annales*

---

[*] Amer. Journ., Vol. 27, pp. 113-172.

de L'Ecole Normale, 1901), and are able to find the expressions for the cartesian coordinates of all these surfaces.

When the generalization of the Bäcklund transformation of pseudospherical surfaces is a generalization of the transformation of Bianchi of the latter, we call it the *complementary* transformation (see §2). In §6 we find that every complementary transform of a moulure surface has plane lines of curvature in one system; that every surface with plane lines of curvature in one system has two at most of such transforms; that this is true of no other A-surfaces.

In §7 we find the conditions to be satisfied in order that the curves $v =$ const. be spherical and certain particular cases are discussed, including surfaces of Joachimstal which are of this class also. In §8 it is given to find the character of an A-surface in order that it may admit of a complementary transform with spherical lines of curvature in one system. And, finally, it is shown that there are an infinity of A-surfaces, each of which admits of an infinity of transforms with spherical lines of curvature in one system.

§2. *General Formulae.*

The parameters of the lines of curvature upon a pseudospherical surface of unit curvature can be so chosen that the linear element of the surface can be written

$$(1) \qquad ds^2 = \cos^2 \omega du^2 + \sin^2 \omega dv^2,^*$$

where ω is a solution of the equation

$$(2) \qquad \frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega ;$$

and every solution of this equation determines a pseudospherical surface. The linear element of the spherical representation is

$$(3) \qquad d\sigma^2 = \sin^2 \omega du^2 + \cos^2 \omega dv^2.^*$$

This evidently is the expression for the linear element of the spherical representation of the lines of curvature of certain A-surfaces, and each solution ω of equation (2) gives rise to a whole group of these. For if the linear element of such a surface be written

$$(4) \qquad ds^2 = A^2 du^2 + C^2 dv^2,$$

---

* Darboux, Leçons, t. III, p. 377.

the coefficients A and C must satisfy the conditions

$$(5)\qquad -\frac{1}{C}\frac{\partial A}{\partial v}=\frac{\partial \omega}{\partial v},\quad \frac{1}{A}\frac{\partial C}{\partial u}=\frac{\partial \omega}{\partial u},$$

and every set of functions A and C satisfying these equations determines an A-surface. We have shown* that the solution of the system (5) requires the integration of a Laplace equation, from which one gets an idea of the number of A-surfaces in a *group* with the same spherical representation.

Consider the system of equations

$$(6)\qquad \begin{cases} \dfrac{\partial \theta}{\partial u}+\dfrac{\partial \omega}{\partial v}=\cos\omega\sin\theta, \\ \dfrac{\partial \theta}{\partial v}+\dfrac{\partial \omega}{\partial u}=-\sin\omega\cos\theta. \end{cases}$$

The integration of this system is reducible to the solution of a Riccati equation; consequently the general integral involves an arbitrary constant. Given a particular integral θ_1; in the tangent planes to an A-surface with the spherical representation (3), draw the lines through the points of contact, making the angle θ_1 with the lines of curvature $v=$ const. We have shown† that the planes perpendicular to the tangent planes and meeting them in these lines envelope a new A-surface S_1 with the lines of curvature parametric, and for which the linear element of the spherical representation is

$$(7)\qquad d\sigma_1^2=\sin^2\theta_1 du^2+\cos^2\theta_1 dv^2\,;$$

and the coefficients of the linear element of S_1 have the expression

$$(8)\qquad \begin{cases} A_1=\dfrac{\partial A}{\partial u}+C\dfrac{\partial \omega}{\partial u}+\cos\theta_1(A\cos\omega+C\sin\omega), \\ C_1=\dfrac{\partial C}{\partial v}-A\dfrac{\partial \omega}{\partial v}+\sin\theta_1(A\cos\omega+C\sin\omega). \end{cases}$$

We call this transformation *complementary;* when the given surface is pseudospherical it is the transformation discovered by Bianchi and called after him by Darboux.

§3. *A-Surfaces with Plane Lines of Curvature.*

In order that the lines of curvature $v=$ const. on a surface be plane it is necessary and sufficient that the spherical representation of these lines be small circles, and consequently that the lines have constant geodesic curvature. The general expression for the geodesic curvature of the lines $v = \text{const.}$ is*

---

*Amer. Journ., l. c. p. 118.

†Amer. Journ., l. c. p. 129.

$$\frac{1}{\rho_v} = -\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v},$$

where E and G have their customary significance. Hence, if the given surface is an A-surface with the spherical representation (3), the condition that the curves $v = \text{const.}$ on the sphere be circles is

$$\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v} = V, \tag{9}$$

where V denotes a function of v alone. In a similar manner the necessary and sufficient condition that the lines $u = \text{const.}$ be plane is

$$\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u} = U, \tag{10}$$

where U denotes a function of u alone. If these conditions are to be satisfied simultaneously, we must have

$$\frac{\partial}{\partial u}(\sin\omega \,.\, V) = \frac{\partial}{\partial v}(\cos\omega U),$$

which reduces by means of (9) and (10) to

$$U.V = 0.$$

Assume that $V = 0$, then it follows from (9) that ω is a function of u alone. We have shown† that this is the necessary and sufficient condition that the corresponding A-surfaces be moulure surfaces. Hence *moulure surfaces are the only A-surfaces with plane lines of curvature in both systems.*

We limit ourselves now to the case where only the lines $v = \text{const.}$ are plane. Equation (9) must be satisfied, which may be written in either of the forms

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right) = 0, \quad \frac{\partial}{\partial v}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u}\right) = 0. \tag{11}$$

Write these equations in the form

$$\frac{\partial\omega}{\partial u} = U\sin\omega, \quad \frac{\partial\omega}{\partial v} = V\sin\omega \tag{12}$$

---

*Bianchi, Lezioni I, 181.

†Amer. Journ., l. c. p. 122.

and substitute in equation (2); by means of (12) the result may be reduced to

$$\cos\omega = \frac{V' - U'}{U^2 - (V^2 + 1)}. \tag{13}$$

When this value of ω is substituted in the first of (12), it becomes

$$UU'' - \frac{U''}{U} - U'^2 - (U^2 - 1)^2 - V^2\frac{U''}{U} + 2(U^2 - 1)V^2 + V'^2 - V^4 = 0; \tag{14}$$

in like manner the second of (12) becomes

$$VV'' + \frac{V''}{V} - V'^2 - (U^2 - 1)^2 - U^2\frac{V''}{V} + 2(U^2 - 1)V^2 + U'^2 - V^4 = 0. \tag{15}$$

Differentiate (14) with respect to v; this gives

$$V'\left[V\frac{U''}{U} - 2(U^2 - 1)V + 2V^3 - V''\right] = 0. \tag{16}$$

If V' is zero, ω as given by (13) is a function of u alone, in which case the corresponding surfaces are moulure surfaces, and consequently the lines of curvature are plane in both systems. Excluding this case, equation (16) may be written

$$\frac{U''}{U} - 2(U^2 - 1) + 2V^2 - \frac{V''}{V} = 0,$$

which may be replaced by the two equations

$$\frac{U''}{U} - 2(U^2 - 1) = a, \quad \frac{V''}{V} - 2V^2 = a, \tag{17}$$

where a denotes a constant. From the first of these we have

$$U'^2 = U^4 + (a - 2)U^2 + b \tag{18}$$

where b is a constant. When this value is substituted in (14), one gets

$$V'^2 = V^4 + aV^2 + (a + b - 1). \tag{19}$$

This value of V is seen to be consistent with the second of (17). And these values of U and V determined by these equations satisfy equation (15). Hence every solution of equations (18) and (19) gives a configuration upon the sphere which serves for the spherical representation of the lines of curvature of A-surfaces with the lines $v =$ const. plane. As a and b are arbitrary constants, there is a double infinity of these systems and they are determined by two quadratures, as is seen from the form of (18) and (19).

§4. *A-Surfaces of Joachimstal.*

Surfaces with plane lines of curvature in one system, such that all the planes pass through a line L, are called surfaces of Joachimstal. We inquire whether any of these surfaces are A-surfaces.

The necessary and sufficient condition that a surface S with plane lines of curvature in one family be a surface of Joachimstal is that the lines of curvature in the other system lie on spheres which cut the surface orthogonally; in this case the centres of the spheres lie on the line L*. If a curve C on S is the intersection of S and a sphere, which cuts the surface orthogonally, the radii of the sphere to the points of C form the developable circumscribing S along C, and the centre of the sphere, since it is the vertex of the cone, is the centre of the geodesic curvature of C, by the theorem of Beltrami;† evidently this curvature is constant. Conversely, Brioschi has shown that when a line of curvature has constant geodesic curvature, it is the intersection of the surface and a sphere cutting the surface orthogonally.‡ Hence our problem reduces to the determination of the A-surfaces with the lines $v=$ const. plane, and with the lines of curvature $u=$ const. of constant geodesic curvature.

In consequence of (5) the geodesic curvature of the lines $u=$ const. is given by§

$$\frac{1}{\rho_u}=-\frac{1}{AC}\frac{\partial C}{\partial u}=-\frac{1}{C}\frac{\partial \omega}{\partial u},$$

hence for these lines to be of constant geodesic curvature we must have

$$\frac{\partial}{\partial v}\left(\frac{1}{C}\frac{\partial \omega}{\partial u}\right)=0.$$

At the same time equations (11) must obtain; consequently we must have

$$C=U_1\sin\omega, \tag{20}$$

where U_1 is any function whatever of u alone. From the second of equations (5) we find that

$$A=U_1\cos\omega+\frac{\sin\omega\,.\,U_1'}{\dfrac{\partial\omega}{\partial u}}, \tag{21}$$

---

* Bianchi, Lezioni, Vol. II, p. 269.

† Bianchi, Ib., Vol. I, p. 277.

‡ Bianchi, Lezioni, Vol. II, p. 296.

§ Bianchi, Lezioni, I, p. 181.

where the accent denotes differentiation with respect to u. In consequence of equations (11) these expressions for A and C satisfy the first of equations (5). Hence we have the theorem:

For each value of ω *given by* (13) *there is a family of A-surfaces of the Joachimstal type, depending upon an arbitrary function.*

Their linear element is of the form

$$ds^2 = \left(U_1 \cos \omega + \frac{U_1'}{U}\right)^2 du^2 + U_1^2 \sin^2 \omega \, dv^2, \tag{22}$$

where U_1 is arbitrary and U is determined by (18). When U_1 is equal to unity, (22) reduces to the form (1). We remarked that for every pseudospherical surface whose curvature is equal to -1 the linear element can be given this form; and when the curvature is another constant the linear element is reducible to the form (1) multiplied by the constant. Hence pseudospherical surfaces with plane lines of curvature in one system are surfaces of Joachimstal. This result was obtained first by Enneper,* and on this account pseudospherical surfaces of this kind are called *surfaces of Enneper.*

From another point of view we shall be able to find the coordinates of these surfaces in finite form. For this we remark that Raffy† has shown that any surface of Joachimstal can be defined by the equations

$$\frac{x}{\cos v} = \frac{y}{\sin v} = \frac{1}{U' \cosh(U + V)}, \quad z = u - \frac{\sinh(U + V)}{U' \cosh(U + V)}, \tag{23'}$$

where U and V are functions of u and v respectively, and the curves $v = \text{const.}$ are plane lines of curvature and $u = \text{const.}$ their orthogonal trajectories. One finds for the linear element of the spherical representation of the surface, as thus defined,

$$d\sigma^2 = \frac{\sin^2 \mu}{\cosh^2(U + V)} dU^2 + \left[\frac{\sinh(U + V)}{\cosh(U + V)} - \mu'\right]^2 dv^2,$$

where μ is a function of v alone defined by

$$\cot \mu = V'.$$

If we put $U = u_1$, or write it $u = U_1$ where U_1 is such a function of u_1 that when u in U is replaced by U_1 it becomes u_1, the equations of the surface become

---

[*] Gött. Nachrichten, 1868, p. 258.

[†] Sur les surfaces à lignes de courbure planes, etc., Ann. L'Ecole Normale, ser. III, Vol. 18, p. 357 (1901).

(on removing the subscript from u_1)

$$\frac{x}{\cos v} = \frac{y}{\sin v} = \frac{U_1'}{\cosh(u+V)}, \quad z = U_1 - U_1' \frac{\sinh(u+V)}{\cosh(u+V)}, \tag{23}$$

where the prime denotes differentiation with respect to u. Now the expression for the linear element on the sphere is

$$d\sigma^2 = \frac{\sin^2 \mu}{\cosh^2(u+V)} du^2 + \left[\frac{\sinh(u+V)}{\cosh(u+V)} - \mu'\right]^2 dv^2. \tag{24}$$

From the form of this expression it follows that V determines the configuration on the sphere, and the character of U_1 has no effect.

Darboux* has given the following general method of generating a surface of Joachimstal: In a plane which passes through a line L one draws any family of circles with centres on L, and also their orthogonal trajectories; if the plane is made to turn about L according to any given law whatever, as one passes from one trajectory to the other, the locus of the trajectories is a surface of Joachimstal. From this statement and the preceding remark concerning (24) it follows that U_1 determines the array of circles, and V the motion of the plane. For our problem U_1 is arbitrary, and V must be determined so that (24) is the representation of the lines of curvature of a pseudospherical surface.

In order to do this we seek the forms of the functions U and V so that equations (23′) shall define a surface of Enneper. When a surface is defined by (23′), its principal radii of curvature are given by

$$\left\{\begin{aligned} \rho_1 &= \frac{\sqrt{1+V'^2}}{U'}\left[\sinh(U+V) + \frac{U''}{U'^2}\cosh(U+V)\right], \\ \rho_2 &= \frac{-\sqrt{1+V'^2}}{U'\left[\sinh(U+V) + \frac{V''}{1+V'^2}\cosh(U+V)\right]} \end{aligned}\right. \tag{25}$$

From these expressions we find that the necessary and sufficient condition that the curvature of the surface be -1 is

$$\sinh(U+V)\left[1 - \frac{U'^2}{1+V'^2}\right] + \cosh(U+V)\left[\frac{U''}{U'^2} - \frac{V''U'^2}{(1+V'^2)^2}\right] = 0. \tag{26}$$

From (25) it is seen that $U' \neq 0$. We consider first the case where $V' = 0$; from the form of the foregoing expressions it is seen that in all generality we may take $V = 0$, but then equations (23′) define surfaces of revolution which are an evident solution of the problem, and will be excluded from the further discussion. With this understanding we put

---

* Leçons, Vol. 1, p. 117.

$$(27)\qquad U_2 = \frac{1}{U'^2}, \quad V_2 = \frac{1}{1 + V'^2}$$

and take U and V for independent variables, designating differentiation with respect to them by accents. Equation (26) reduces to

$$(28)\qquad 2\tanh(U + V)(U_2 - V_2) = U_2' - V_2'.$$

Differentiate this equation with respect to V, and multiply by $\cosh^2(U + V)$; this gives

$$2(U_2 - V_2) - V_2' \sinh 2(U + V) + V_2'' \cosh^2(U + V) = 0.$$

If this equation be differentiated with respect to V, the result reduces to

$$V_2''' = 4V_2',$$

from which it follows that

$$(29)\qquad V_2 = a \cosh 2V + b \sinh 2V + c,$$

where a, b, c are constants. From the form of (28) it follows that in like manner

$$(29')\qquad U_2 = a_1 \cosh 2U + b_1 \sinh 2U + c_1.$$

When these values are substituted in (27) we find

$$a_1 = -a, \quad b_1 = b, \quad c_1 = c.$$

From (27) and (29) it follows that the function V determining a system of lines on the sphere which serve for the representation of the lines of curvature of an A-surface of Joachimstal is given by the quadrature

$$(30)\qquad \int \sqrt{\frac{a \cosh 2V + b \sinh 2V + c}{-a \cosh 2V - b \sinh 2V - (c - 1)}}\, dV = v.$$

When the expression for V from this equation is substituted in (23′) or (23) they define all the A-surfaces of Joachimstal. When the parameter u has been chosen so that the equations take the form (23), the form of U_1 for the corresponding surface of Enneper is given by the quadrature

$$U_1 = \int \sqrt{c - a \cosh 2u + b \sinh 2u}\,.\,du,$$

in consequence of (27) and (29′).

The form of (18) is not altered if u and V be replaced by $u + x$ and $V - x$ respectively, where x is a constant which can be so chosen as to remove the term $\cosh 2V$ or $\sinh 2V$ from (29). Hence only two of the constants a, b, c are arbitrary, so that there is on the sphere only a double infinity of the curves sought, as was found before.

§5. *A-surfaces with Plane Lines of Curvature in One System.*

We proceed now to the determination of the A-surfaces with plane lines of curvature in one system which are not necessarily surfaces of Joachimstal. We remark that the planes of these lines of curvature are parallel to the corresponding planes of the lines of curvature of the pseudospherical surface with the same spherical representation as the given surface. But, as we have seen, the planes for the pseudospherical surface pass through a line L, and consequently for the given surface they either pass through a line or envelope a cylinder whose elements are parallel to L. We have considered the former in the preceding section and now will turn to the latter. For this the recent investigations by Raffy* will be of great service. He remarks that if S is a surface with plane lines of curvature $v = \text{const.}$, and S_1 is the locus of the centres of normal curvature of these lines, the geodesics $v = \text{const.}$ on S_1 are the curves of contact of circumscribing cylinders, and the angle between the parametric lines on S_1 is a function of v. When, in particular, the planes are parallel to a line L the circumscribing cylinders are parallel to a plane normal to L and conversely. Taking the z-axis for the line L, Raffy† finds, for the surfaces for which the circumscribing cylinders are parallel to the xy plane, expressions which can be put in the form

$$(31)\quad \left\{\begin{aligned} x_1 &= (U_1' \cos v + U_1'' \sin v \cot \mu) \cosh (u + V) \\ &\qquad - (U_1'' \cos v + U_1' \sin v \cot \mu) \sinh (u + V) + \psi \cos v - \psi' \sin v, \\ y_1 &= (U_1' \sin v - U_1'' \cos v \cot \mu) \cosh (u + V) \\ &\qquad - (U_1'' \sin v - U_1' \cos v \cot \mu) \sinh (u + V) + \psi \sin v + \psi' \cos v, \\ z_1 &= U_1 - U_1'', \end{aligned}\right.$$

where U_1 is an arbitrary function of u, ψ an arbitrary function of v; μ denotes the angle between the parametric lines on S_1 and $\cot \mu = V'$, the function V depending only upon v and, as we have seen, determining the spherical representation for S.

From (31) one finds the direction-cosines of the tangents to the curves $v = \text{const.}$ on S_1; as these are the normals to S, we denote them by X, Y, Z, and find for their values

---

*Sur les surfaces a lignes de courbure planes, etc. Ann. L'Ecole Normale, ser. III, Vol. 18, p. 341.

†Ib., p. 352.

$$(32)\quad \begin{cases} X = \cos v \sin \mu \tanh(u + V) - \sin v \cos \mu, \\ Y = \sin v \sin \mu \tanh(u + V) + \cos v \cos \mu, \\ Z = \dfrac{\sin \mu}{\cosh(u + V)}. \end{cases}$$

The surface S is defined by

$$(33)\qquad x = x_1 - \lambda X, \quad y = y_1 - \lambda Y, \quad z = z_1 - \lambda Z,$$

where λ is determined by the condition

$$\Sigma dx_1 . X - d\lambda = 0,$$

or

$$(34)\qquad \frac{\partial \lambda}{\partial u} = \Sigma X \frac{\partial x_1}{\partial u}, \quad \frac{\partial \lambda}{\partial v} = \Sigma X \frac{\partial x_1}{\partial v}.$$

The first of these may be written

$$\frac{\partial \lambda}{\partial u} = \sqrt{E_1} = \frac{(U_1' - U_1''') \cosh(u + V)}{\sin \mu},$$

from which we get

$$(35)\qquad \lambda = \frac{U_1' \sinh(u + V) - U_1'' \cosh(u + V)}{\sin \mu} + \chi(v),$$

where χ is a function of v alone. The second of (34) reduces, in consequence of the definition of μ, to

$$\frac{\partial \lambda}{\partial v} = \cos \mu \sqrt{G_1} = \frac{\cos \mu}{\sin^2 \mu} [(U_1' + \mu' U_1'') \cosh(u + V) - (U_1' \mu' + U_1'') \sinh(u + V)] \\ + [\psi(v) + \psi''(v)] \cos \mu.$$

When the value of λ from (35) is substituted in this equation, it is found that the function $\chi(v)$ is determined by

$$(36)\qquad \chi'(v) = [\psi(v) + \psi''(v)] \cos \mu.$$

From (33), (32), (31) and (35) we have for the coordinates of any A-surface with the lines $v =$ const. plane

$$(37)\quad \begin{cases} x = \left[\dfrac{U_1' - \chi \sin \mu \sinh(u + V)}{\cosh(u + V)} + \psi \cos v\right] \cos v + [\chi \cos v - (\psi \cos v)'] \sin v, \\ y = \left[\dfrac{U_1' - \chi \sin \mu \sinh(u + V)}{\cosh(u + V)} + \psi \cos v\right] \sin v - [\chi \cos v - (\psi \cos v)'] \cos v, \\ z = U_1 - \dfrac{U_1' \sinh(u + V) + \chi \sin \mu}{\cosh(u + V)}, \end{cases}$$

where V is determined by the quadrature (30), U_1 and $\psi(v)$ are arbitrary and $\chi(v)$ is given by (36). For $\psi = 0$ and consequently $\chi = 0$, formulæ (37) reduce to (23), that is, $\psi = 0$ is the necessary and sufficient condition that they be surfaces of Joachimstal. From (37) we derive the following expressions for the coefficients of the linear element of S

$$(38) \qquad \begin{cases} E = \left[\dfrac{\sinh(u+V)\,.\,U_1' + \chi \sin\mu}{U_1' \cosh(u+V)} - \dfrac{U_1''}{U_1'}\right]^2, \\[2ex] G = \left[\dfrac{U_1'}{\sin\mu \cosh(u+V)} - \chi\,\{\tanh(u+V) - \mu'\} + \chi' \tan\mu\right]^2. \end{cases}$$

In particular, when $a = b = 0$ in (30) it reduces to

$$V = \sqrt{\frac{1-c}{c}}\,v + \text{const.}$$

Hence if the parameters are real, $0 < c < 1$. Now

$$\cot\mu = 1 + V'^2 = \frac{1}{C},$$

hence the parametric lines on S_1 cut one another under constant angle.

§6. *Complementary Transformations of Certain A-Surfaces.*

Consider an A-surface S with the spherical representation of its lines of curvature given by (3). Let S_1 be a complementary transform corresponding to a particular integral θ of the system (6). Further we assume that the lines of curvature $v = \text{const.}$ on S_1 are plane; for this we must have (11)

$$\frac{\partial}{\partial v}\left(\frac{1}{\sin\theta}\,\frac{\partial\theta}{\partial u}\right) = 0.$$

By means of (6) this equation may be replaced by

$$(39) \qquad \sin\theta\,\frac{\partial^2\omega}{\partial v^2} + \cos\theta\,\frac{\partial\omega}{\partial u}\,\frac{\partial\omega}{\partial v} + \sin\omega\,\frac{\partial\omega}{\partial v} = 0.$$

This equation is satisfied identically when $\dfrac{\partial\omega}{\partial v} = 0$, that is, when S is a moulure surface. Hence:

The complementary transforms of a moulure surface have plane lines of curvature in one system.

From (39) it is seen that, if S is not a moulure surface, two at most of its complementary transforms have plane lines of curvature in one system. The angle θ giving these transforms is obtained from (39); but this function must satisfy (6) also. Instead of solving (39) and substituting in (6), we differentiate the former with respect to v and take account of the latter. This gives

$$\sin\theta\left[\frac{\partial^3\omega}{\partial v^3}+\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v}-\sin^2\omega\,\frac{\partial\omega}{\partial v}\right]+\cos\theta\,\frac{\partial\omega}{\partial v}\,\frac{\partial^2\omega}{\partial u\partial v}+\cos\omega\left(\frac{\partial\omega}{\partial v}\right)^2=0.$$

The values of θ given by this equation must be the same as in (39), hence we must have

$$\text{(40)}\qquad \frac{\dfrac{\partial^3\omega}{\partial v^3}+\left(\dfrac{\partial\omega}{\partial u}\right)^2\dfrac{\partial\omega}{\partial v}-\sin^2\omega\,\dfrac{\partial\omega}{\partial v}}{\dfrac{\partial^2\omega}{\partial v^2}}=\frac{\dfrac{\partial^2\omega}{\partial u\partial v}}{\dfrac{\partial\omega}{\partial u}}=\frac{\cos\omega\,\dfrac{\partial\omega}{\partial v}}{\sin\omega}.$$

From the last two we have

$$\text{(41)}\qquad \frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\,\frac{\partial\omega}{\partial v}\right)=0,$$

that is, the lines of curvature $v=$ const. on S are plane. In consequence of (2), the first two ratios of (40) are equal when (41) is satisfied. In like manner we differentiate (39) with respect to u and take account of (6); this gives

$$\sin\theta\left[\frac{\partial^3\omega}{\partial u\partial v^2}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\right]+\cos\theta\,\frac{\partial\omega}{\partial u}\,\frac{\partial^2\omega}{\partial u\partial v}+\sin\omega\,\frac{\partial^2\omega}{\partial u\partial v}=0,$$

so that ω must satisfy

$$\frac{\partial\omega}{\partial v}\,\frac{\partial^3\omega}{\partial u\partial v^2}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3-\frac{\partial^2\omega}{\partial u\partial v}\,\frac{\partial^2\omega}{\partial v^2}=0.$$

But this equation also is a consequence of (41). Hence:

If the lines $v=$ const. on an A-surface are plane, the two values of θ given by (39) *determine complementary transformations, such that the lines $v=$ const. on the transforms are plane, and on all the other transforms the lines $v=$ const. are skew.*

An exception to this theorem arises for the case where the two values of θ, given by (39), coincide. The necessary and sufficient condition for this is

$$\text{(42)}\qquad \left(\frac{\partial^2\omega}{\partial v^2}\right)^2+\left[\left(\frac{\partial\omega}{\partial u}\right)^2-\sin^2\omega\right]\left(\frac{\partial\omega}{\partial v}\right)^2=0,$$

and the single value is given by

$$\text{(43)}\qquad \cos\theta=-\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\sin\omega.$$

Suppose we have given a function ω, satisfying (42), and we denote by S a surface with the spherical representation of its lines of curvature determined by ω. Let S_0 be a surface of which S is the complementary transform by means of ω, and let its spherical representation be given by

$$d\sigma_0^2 = \sin^2\phi du^2 + \cos^2\phi dv^2;$$

then ϕ is a particular integral of the system

$$\frac{\partial\omega}{\partial u} + \frac{\partial\phi}{\partial v} = \sin\omega\cos\phi, \quad \frac{\partial\omega}{\partial v} + \frac{\partial\phi}{\partial u} = -\cos\omega\sin\phi. \tag{44}$$

In consequence of the above results we know that the lines $v = \text{const.}$ on S_0 are plane, so that ϕ must satisfy

$$\frac{\partial}{\partial v}\left(\frac{1}{\sin\phi}\frac{\partial\phi}{\partial u}\right) = 0. \tag{45}$$

By means of (45) and (44) equation (42) may be replaced by

$$\operatorname{cosec}^2\phi \,.\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial\phi}{\partial u} + \cos\omega\sin\phi\right)^2 = 0,$$

which reduces to

$$\frac{\partial\phi}{\partial v}\cdot\frac{\partial\omega}{\partial v} = 0.$$

If ω were a function of u alone, all the transforms of S would have plane lines of curvature $v = \text{const.}$ Hence the above equation gives $\phi = \phi(u)$, and consequently S_0 is a moulure surface. We have then the theorem:

If a surfacc S is a complementary transform of a moulure surface, it admits only one complementary transformation giving rise to a surface with plane lines of curvature in one system; the determining function θ of the transformation is given by (43); *for all other A-surfaces with plane lines of curvature there are two such transformations.*

§7. *A-Surfaces with Spherical Lines of Curvature.*

Enneper* has shown that the necessary and sufficient condition that the lines of curvature $v = \text{const.}$ on a surface be spherical is that the following relation obtains

$$A = R\cos\sigma\frac{A}{\rho_1} + R\sin\sigma\,\frac{\rho_2}{C}\frac{\partial}{\partial v}\frac{A}{\rho_1},$$

where R is the radius of the sphere and σ the angle under which the latter cuts the surface; both of these quantities are functions of v alone.

---

* Gött. Nachrichten, 1868, p. 421.

For an A-surface

$$A = -\rho_1 \sin\omega, \qquad C = \rho_2 \cos\omega,$$

so that for these surfaces the equation of condition is

$$A = \alpha \sin\omega + \beta \frac{\partial\omega}{\partial v}, \tag{46}$$

where α and β are functions of v alone.

Consider first the case where the surface is pseudospherical; equation (46) becomes

$$\cot\omega = \alpha + \frac{\beta}{\sin\omega}\frac{\partial\omega}{\partial v}.$$

Differentiate with respect to u; this gives

$$\frac{1}{\sin^2\omega}\frac{\partial\omega}{\partial u} + \beta\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right) = 0.$$

Differentiate again with respect to u and eliminate β. This gives the following condition to be satisfied by ω:

$$\begin{vmatrix} \dfrac{1}{\sin^2\omega}\dfrac{\partial\omega}{\partial u} & \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sin\omega}\dfrac{\partial\omega}{\partial v}\right) \\ \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sin^2\omega}\dfrac{\partial\omega}{\partial u}\right) & \dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{\sin\omega}\dfrac{\partial\omega}{\partial v}\right) \end{vmatrix} = 0, \tag{47}$$

which reduces to

$$\frac{\partial\omega}{\partial u}\frac{\partial^3\omega}{\partial u^2\partial v} - \frac{\partial^2\omega}{\partial u^2}\frac{\partial^2\omega}{\partial u\partial v} + \left(\frac{\partial\omega}{\partial u}\right)^3\frac{\partial\omega}{\partial v} = 0. \tag{48}$$

It is evident that equation (47) is satisfied when

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right) = 0;$$

hence when equation (48) is satisfied the curves $v =$ const. on the pseudospherical surface are *spherical or plane.* In like manner when ω satisfies the equation

$$\frac{\partial\omega}{\partial v}\frac{\partial^3\omega}{\partial u\partial v^2} - \frac{\partial^2\omega}{\partial v^2}\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3 = 0, \tag{49}$$

the lines of curvature $u =$ const. on the corresponding pseudospherical surface are spherical, unless

$$\frac{\partial}{\partial u}\left(\frac{1}{\cos\omega}\frac{\partial\omega}{\partial v}\right) = 0, \tag{50}$$

in which case they are plane. These results have been found by Bianchi.*

---

*Aggiunte alla Memoire "Sopra i sistemi tripli, etc." Annali di Mat., Vol. 14, p. 116 (1887).

We pass to the consideration of A-surfaces in general with one family of spherical lines of curvature. From (46) and the first of equations (5) we have

$$C = -\left(\frac{\alpha' \sin \omega + \beta \dfrac{\partial^2 \omega}{\partial v^2}}{\dfrac{\partial \omega}{\partial v}} + \beta' + \alpha \cos \omega \right), \tag{51}$$

where the accent denotes differentiation with respect to v. When this expression for C is substituted in the second of equations (5), the latter becomes

$$\alpha' \left[\cos \omega \frac{\partial \omega}{\partial u} \frac{\partial \omega}{\partial v} - \sin \omega \frac{\partial^2 \omega}{\partial u \partial v} \right] + \beta \left[\frac{\partial \omega}{\partial v} \frac{\partial^3 \omega}{\partial u \partial v^2} - \frac{\partial^2 \omega}{\partial v^2} \frac{\partial^2 \omega}{\partial u \partial v} + \frac{\partial \omega}{\partial u} \left(\frac{\partial \omega}{\partial v} \right)^3 \right] = 0. \tag{52}$$

If the expression in the first parenthesis is zero, the same is true of the second; as this is the condition that the lines $v =$ const. be plane, it may be neglected.

Another solution of this equation is obtained by taking α' equal to zero, and also the expression in the second parenthesis. The vanishing of the latter is the condition (49) that the lines $u =$ const. be plane or spherical on the pseudospherical surface, having the same spherical representation as the given surface; hereafter we shall say that this pseudospherical surface is *associated* with the given surface. Two cases are to be considered according as α is equal to or different from zero; in the latter case it may be taken equal to unity.

1° $\alpha = 0$. From the definition of α and β it follows that in this case σ is a right-angle and β is the expression for the radius of the sphere; hence when β is constant the spheres upon which the curves $v =$ const. lie are spherical. The linear element of the surface is

$$ds^2 = \beta^2 \left(\frac{\partial \omega}{\partial v} \right)^2 du^2 + \left(\beta \frac{\partial^2 \omega}{\partial v^2} + \beta' \frac{\partial \omega}{\partial v} \right)^2 \frac{dv^2}{\left(\dfrac{\partial \omega}{\partial v} \right)^2}, \tag{53}$$

where β is an arbitrary function of v alone.

2° $\alpha = 1$. The projection of the radius of the sphere on the normal to the surface is constant. All the spheres have the same radius when β is constant, in which case the spheres and the surface cut under constant angle and only in this case. The linear element of these surfaces is

$$ds^2 = \left(\sin \omega + \beta \frac{\partial \omega}{\partial v} \right)^2 du^2 + \left(\cos \omega + \beta' + \beta \frac{\dfrac{\partial^2 \omega}{\partial v^2}}{\dfrac{\partial \omega}{\partial v}} \right) dv^2. \tag{54}$$

Gathering together these results we have the theorem:

Given a pseudospherical surface with lines of curvature $u = \text{const.}$ plane or spherical; among all the A-surfaces with which it is associated there are two families, each containing an infinity of surfaces whose lines of curvature $v = \text{const.}$ are spherical.

When the lines $u = \text{const.}$ on the pseudospherical surface are plane, they are plane for all the A-surfaces associated with it. Now

$$\frac{\partial\omega}{\partial v} = \cos\omega \,.\, V,$$

where V is a function of v alone. If we substitute this value in (53), it becomes

$$ds^2 = V_1^2 \cos^2 \omega du^2 + \left(\frac{V_1'}{V} - V_1 \sin\omega\right)^2 dv^2, \tag{55}$$

where we have put $V_1 = \beta V$, so that V_1 is arbitrary and V is determined in a manner similar to U in §3. If (55) is compared with (22), it will be seen that the former defines the linear element of the A-surfaces of Joachimstal with $u = \text{const.}$ plane lines of curvature. In a similar manner we have from (54) in this special case

$$ds^2 = (V_1 \cos\omega + \sin\omega)^2 du^2 + \left(\cos\omega - V_1 \sin\omega + \frac{V_1'}{V}\right)^2 dv^2. \tag{56}$$

From this we have the theorem:

In addition to the family of surfaces of Joachimstal associated with a pseudospherical surface of Enneper there is another infinite family of A-surfaces with plane lines of curvature in one system and spherical lines in the other. Their linear element is of the form (56).

Excluding these particular solutions of (52), we differentiate it with respect to u and eliminate α' and β, with the result

$$\begin{vmatrix} \cos\omega \dfrac{\partial\omega}{\partial u}\dfrac{\partial\omega}{\partial v} - \sin\omega \dfrac{\partial^2\omega}{\partial u \partial v}\,, & \dfrac{\partial\omega}{\partial v}\dfrac{\partial^3\omega}{\partial u \partial v^2} - \dfrac{\partial^2\omega}{\partial v^2}\dfrac{\partial^2\omega}{\partial u \partial v} + \dfrac{\partial\omega}{\partial u}\left(\dfrac{\partial\omega}{\partial v}\right)^3 \\ \dfrac{\partial}{\partial u}\left[\cos\omega \dfrac{\partial\omega}{\partial u}\dfrac{\partial\omega}{\partial v} - \sin\omega \dfrac{\partial^2\omega}{\partial u \partial v}\right], & \dfrac{\partial}{\partial u}\left[\dfrac{\partial\omega}{\partial v}\dfrac{\partial^3\omega}{\partial u \partial v^2} - \dfrac{\partial^2\omega}{\partial v^2}\dfrac{\partial^2\omega}{\partial u \partial v} + \dfrac{\partial\omega}{\partial u}\left(\dfrac{\partial\omega}{\partial v}\right)^3\right] \end{vmatrix} = 0. \tag{57}$$

When a solution of this equation is found and substituted in (52), α' can be chosen arbitrarily, and β is entirely determined; the coefficients of the linear element are given by (46) and (51). Hence the theorem:

Given a solution ω of equation (57) *which satisfies* (2); *this function serves to determine the spherical representation of the lines of curvature of an infinity of A-surfaces on which the lines of curvature $v = \text{const.}$ are spherical.*

When ω satisfies equation (49), it is a solution of (57), consequently we have the theorem:

When the lines of curvature $u = \text{const.}$ on a pseudospherical surface are spherical, there is an infinity of A-surfaces associated with it whose lines $v = \text{const.}$ are spherical; and the fundamental coefficients are given without quadrature.

Moreover, when ω satisfies (50) it is a solution not only of (49), but of (48); this can be shown by differentiating (50) with respect to u. Hence the theorem:

When the lines of curvature in one system on an A-surface are plane, the lines of curvature in the other system are spherical.

§8. *Complementary Transforms with Spherical Lines of Curvature.*

Bianchi has considered* the problem of finding the pseudospherical surfaces whose complementary transforms have spherical lines of curvature in one system. We shall recall his investigation briefly before proceeding to the discussion of the same problem for A-surfaces in general.

Denote by Σ the complementary transform of the pseudospherical surface S and write the linear element of the spherical representation of the former in the form

$$d\sigma^2 = \sin^2\theta du^2 + \cos^2\theta dv^2, \tag{58}$$

recalling that θ and ω are bound by the relations

$$\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v} = \sin\theta\cos\omega, \quad \frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u} = -\cos\theta\sin\omega. \tag{59}$$

For the lines $v = \text{const.}$ on Σ to be spherical θ must satisfy an equation of the form (47), which by means of (59) can be written

$$A\cos\theta + B\sin\theta + C = 0,$$

where we have put

$$A = \sin\omega\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v} + \cos\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v},$$

$$B = \cos\omega\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2 + \cos\omega\frac{\partial^3\omega}{\partial u\partial v^2} + \sin\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial v^2},$$

$$C = \sin^2\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} - \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3 + \sin\omega\cos\omega\frac{\partial^2\omega}{\partial u\partial v} - \frac{\partial\omega}{\partial v}\frac{\partial^3\omega}{\partial u\partial v^2} + \frac{\partial^2\omega}{\partial v^2}\frac{\partial^2\omega}{\partial u\partial v}.$$

---

*l. c. p. 118.

From the above it follows that for more than two of the transforms to have the desired property, it is necessary that A, B and C be zero.

Putting A equal to zero, we find that either $\frac{\partial\omega}{\partial u}$ must be zero or ω be a solution of equation (50). In the former case S is a surface of revolution with the meridians $u = \text{const.}$, and we have noted that the latter is the condition that the curves $u = \text{const.}$ on S be plane. These values make B and C vanish so that we have the theorem of Bianchi:

For all the complementary transforms of a pseudospherical surface S to have spherical lines of curvature $v = \text{const.}$ it is necessary and sufficient that the lines of curvature $u = \text{const.}$ on S be plane, that is, S must be a surface of Enneper. Bianchi has shown that the transforms are surfaces of Enneper only in case S is a surface of revolution.

We proceed now to the general problem of finding the A-surfaces whose complementary transforms have spherical lines of curvature in one system, say $v = \text{const.}$ Let the spherical representation of these lines be given by (58) with the relations (59). Now θ must satisfy an equation of the form (57), which by means of (59) can be reduced to

$$\tag{60} \begin{vmatrix} l \cos\theta + m \sin\theta + n & a \cos\theta + b \sin\theta + c \\ \frac{\partial}{\partial u}(l \cos\theta + m \sin\theta + n) & \frac{\partial}{\partial u}(a \cos\theta + b \sin\theta + c) \end{vmatrix} = 0,$$

where we have put, for the sake of brevity,

$$\tag{61} \left\{ \begin{aligned} & l = \frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v}, \quad m = \frac{\partial^2\omega}{\partial v^2}, \quad n = \sin\omega\frac{\partial\omega}{\partial v}, \\ & a = \sin\omega\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v} + \frac{\partial^3\omega}{\partial u^2\partial v}\sin\omega - \cos\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u^2}, \\ & b = \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\cos\omega - \sin\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u\partial v}, \\ & c = \frac{\partial\omega}{\partial u}\frac{\partial^3\omega}{\partial u^2\partial v} + \frac{\partial\omega}{\partial v}\left(\frac{\partial\omega}{\partial u}\right)^3 - \frac{\partial^2\omega}{\partial u^2}\frac{\partial^2\omega}{\partial u\partial v} - \cos^2\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} + \sin\omega\cos\omega\frac{\partial^2\omega}{\partial u\partial v}. \end{aligned} \right.$$

From the discussion of equation (57) it follows that equation (60) is satisfied when the lines $v = \text{const.}$ on the transforms are plane, that is when

$$\frac{\partial}{\partial v}\left(\frac{1}{\sin\theta}\frac{\partial\theta}{\partial u}\right) = 0.$$

But we have seen that, unless the given surface is a moulure surface, only two of the transforms will have the curves $v = \text{const.}$ plane. Hence when equation

(60) is satisfied for more than two values of θ, the lines $v=$ const. on the corresponding transforms are spherical.

We return now to equation (60) and find that when the determinant is developed the equation can be written

$$A + B\cos\theta + C\sin\theta + D\sin\theta\cos\theta + E\sin^2\theta = 0,$$

where we have put

$$(62)\quad \left\{\begin{aligned}
A &= nc' - n'c + a'l - al' + (am - bl)\frac{\partial\omega}{\partial v},\\
B &= na' - n'a + c'l - cl' + (cm - bn)\frac{\partial\omega}{\partial v},\\
C &= nb' - n'b + c'm - cm' + (an - cl)\frac{\partial\omega}{\partial v} + (bl - am)\cos\omega,\\
D &= lb' - l'b + a'm - am' + (bn - cm)\cos\omega,\\
E &= mb' - m'b + al' - a'l + (cl - an)\cos\omega,
\end{aligned}\right.$$

the primes denoting differentiation with respect to u. From the form of the above equation it is clear that for more than four of the transforms to have spherical lines of curvature, it is necessary that A, B, C, D, E vanish, for we are dealing with real surfaces.

If A and B are equated to zero and the resulting equations are solved for a' and c', one finds

$$(63)\quad \begin{aligned}
&\left(\frac{\partial\omega}{\partial v}\right)^2\left[\left(\frac{\partial\omega}{\partial u}\right)^2 - \sin^2\omega\right]\Bigg\{a' - \left[\left(\frac{\partial\omega}{\partial u}\right)^2 + \frac{\dfrac{\partial^3\omega}{\partial u^2\partial v}}{\dfrac{\partial\omega}{\partial v}}\right]\left(\sin\omega\frac{\partial^2\omega}{\partial u\partial v} + \cos\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v}\right)\\
&\quad + 2\cos\omega\frac{\partial^2\omega}{\partial u\partial v}\frac{\partial^2\omega}{\partial u^2} + \cos^3\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} - \sin\omega\cos^2\omega\frac{\partial^2\omega}{\partial u\partial v} - \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\cos\omega\\
&\quad + \sin\omega\left(\frac{\partial\omega}{\partial v}\right)^2\frac{\partial^2\omega}{\partial u\partial v}\Bigg\} = 0,
\end{aligned}$$

$$(63)\quad \begin{aligned}
&\left(\frac{\partial\omega}{\partial v}\right)^2\left[\left(\frac{\partial\omega}{\partial u}\right)^2 - \sin^2\omega\right]\Bigg\{c' - \left[\left(\frac{\partial\omega}{\partial u}\right)^2 + \frac{\dfrac{\partial^3\omega}{\partial u^2\partial v}}{\dfrac{\partial\omega}{\partial v}}\right]\left(\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v} + \sin\omega\cos\omega\frac{\partial\omega}{\partial v}\right)\\
&\quad - \sin\omega\cos\omega\frac{\left(\dfrac{\partial^2\omega}{\partial u\partial v}\right)^2}{\dfrac{\partial\omega}{\partial v}} + \cos^2\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u^2} + \frac{\dfrac{\partial^2\omega}{\partial u^2}\left(\dfrac{\partial^2\omega}{\partial u\partial v}\right)^2}{\dfrac{\partial\omega}{\partial v}}\\
&\quad + \cos^2\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v}\Bigg\} = 0.
\end{aligned}$$

These equations are satisfied by $\frac{\partial\omega}{\partial v}$ equal to zero, and from the expressions for a, b, c it is clear that equation (60) also is satisfied by this value. But we have seen that for this case the curves $v=$ const. are plane for the transforms, and consequently may be neglected.

Equations (63) are satisfied also when

$$\left(\frac{\partial\omega}{\partial u}\right)^2-\sin^2\omega=0,$$

from which we get

$$\tan\frac{\omega}{2}=u\,.\,V, \tag{64}$$

where V is a function of v alone. From the above equation and (2) we find that $\frac{\partial^2\omega}{\partial v^2}$ is zero, and for ω given by (64) to satisfy this condition V must be a constant; hence this case leads back to the former.

We consider finally the case where the expressions in the brackets in (63) are zero. If the values of a' and c' given in this manner be substituted in the expressions for C and D and the latter be equated to zero, we get

$$\begin{aligned} a\left[m'\frac{\partial\omega}{\partial v}-m\frac{\partial^2\omega}{\partial u\partial v}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\right]&=0,\\ b\left[m'\frac{\partial\omega}{\partial v}-m\frac{\partial^2\omega}{\partial u\partial v}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\right]&=0. \end{aligned} \tag{65}$$

From (61) it follows that if b is equal to zero, we have $\frac{\partial\omega}{\partial v}$ zero, which has been considered, or ω must satisfy (11). When the latter is satisfied, both a and c vanish and equation (60) is satisfied. Hence when the lines $v=$ const. on the surface are plane and consequently $u=$ const. are spherical, the lines $v=$ const. on the transforms, or surfaces with the same spherical representation, are spherical.

There is one further case to be considered, namely, when the expression in the parenthesis of (65) is zero, which reduces readily to (49). When this equation holds, the values of a' and c' obtained from (61) are found to satisfy (63) and make E equal to zero. Hence:

The necessary and sufficient condition that for each solution θ of equations (59) *there is an infinity of surfaces with the lines $v=$ const. spherical is that ω be a solution of equations* (2) *and* (49).

Suppose now that we have a function ω satisfying these conditions. Among all the surfaces with the corresponding representation of the lines of curvature there must be an infinity which admit by means of a solution of equations (59) transforms upon which the lines $v =$ const. are spherical. We inquire whether there are any of them for which the transforms by all the functions θ, given by (59), have spherical lines of curvature $v =$ const.

Denote by A_1 and C_1 the coefficients of the linear element for one of the transforms S_1. Now it must be possible for A_1 and C_1 to assume forms of expression similar to (46) and (51). Hence from (8) the necessary and sufficient condition that all the transforms of the same surface have spherical lines of curvature $v =$ const. is that the following equations be satisfied:

$$(66)\quad \begin{cases} \dfrac{\partial A}{\partial u} + C\dfrac{\partial \omega}{\partial u} + \lambda \cos\theta = \alpha \sin\theta + \beta\dfrac{\partial \theta}{\partial v}, \\[2ex] \dfrac{\partial C}{\partial v} - A\dfrac{\partial \omega}{\partial v} + \lambda \sin\theta = -\dfrac{\dfrac{\partial \alpha}{\partial v}\sin\theta + \beta\dfrac{\partial^2\theta}{\partial v^2} + \left(\dfrac{\partial \beta}{\partial v} + \alpha\sin\theta\right)\dfrac{\partial\theta}{\partial v}}{\dfrac{\partial\theta}{\partial v}} \end{cases}$$

for every value of θ satisfying (59), and the corresponding values of α and β given by

$$(67)\quad \frac{\partial \alpha}{\partial v}\left[\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} - \sin\theta\frac{\partial^2\theta}{\partial u\partial v}\right] + \beta\left[\frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial v^2}\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^3\right] = 0,$$

where we have put

$$(68)\qquad \lambda = A\cos\omega + C\sin\omega.$$

The general solution θ of (59) involves a parameter, say t, which in general will appear in α and β, but not in A, C and ω. If then equations (66) are differentiated with respect to t, they become

$$(69)\quad \begin{aligned} \lambda\sin\theta\frac{\partial\theta}{\partial t} &= -\left[\alpha\cos\theta\frac{\partial\theta}{\partial t} + \sin\theta\frac{\partial\alpha}{\partial t} + \frac{\partial\theta}{\partial v}\frac{\partial\beta}{\partial t} + \beta\frac{\partial^2\theta}{\partial v\partial t}\right], \\ \lambda\cos\theta\frac{\partial\theta}{\partial t} &= -\left[\frac{\sin\theta\dfrac{\partial^2\alpha}{\partial v\partial t} + \cos\theta\dfrac{\partial\alpha}{\partial v}\dfrac{\partial\theta}{\partial t} + \dfrac{\partial\beta}{\partial t}\dfrac{\partial^2\theta}{\partial v^2} + \beta\dfrac{\partial^3\theta}{\partial t\partial v^2}}{\dfrac{\partial\theta}{\partial v}} + \frac{\partial^2\beta}{\partial v\partial t}\right. \\ &\qquad \left. + \frac{\partial\alpha}{\partial t}\cos\theta - \alpha\sin\theta\frac{\partial\theta}{\partial t} - \frac{\dfrac{\partial\alpha}{\partial v}\sin\theta + \beta\dfrac{\partial^2\theta}{\partial v^2}}{\left(\dfrac{\partial\theta}{\partial v}\right)^2}\right]. \end{aligned}$$

Eliminating λ, we are brought to the equation

$$\begin{aligned}
(70)\qquad & -\alpha+\frac{\sin^2\theta}{\frac{\partial\theta}{\partial t}\frac{\partial\theta}{\partial v}}\frac{\partial^2\alpha}{\partial v\partial t}+\frac{\sin\theta}{\left(\frac{\partial\theta}{\partial v}\right)^2}\left(\cos\theta\frac{\partial\theta}{\partial v}-\sin^2\theta\sin\omega\right)\frac{\partial\alpha}{\partial v}\\
&+\frac{1}{\frac{\partial\theta}{\partial t}\frac{\partial\theta}{\partial v}}\left[\sin\theta\frac{\partial^2\theta}{\partial v^2}-\cos\theta\left(\frac{\partial\theta}{\partial v}\right)^2\right]\frac{\partial\beta}{\partial t}+\frac{\sin\theta}{\frac{\partial\theta}{\partial t}}\frac{\partial^2\beta}{\partial v\partial t}\\
&+\frac{\sin\theta}{\left(\frac{\partial\theta}{\partial v}\right)^2}\left[\left(\sin\theta-\cos\omega\frac{\partial\theta}{\partial u}\right)\frac{\partial\theta}{\partial v}-\sin\omega\frac{\partial^2\theta}{\partial v^2}\right]\beta=0.
\end{aligned}$$

Differentiate this equation with respect to u and subtract the two equations; eliminate $\frac{\partial^2\alpha}{\partial v\partial t}$ from this result by means of its value as given by differentiating (67); this gives

$$\begin{aligned}
&\frac{\partial\alpha}{\partial v}-\frac{\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-\sin\theta\frac{\partial^2\theta}{\partial u\partial v}}{\frac{\partial\theta}{\partial t}\frac{\partial\omega}{\partial v}}\frac{\partial^2\beta}{\partial v\partial t}+\left[\sin\theta\frac{\partial^3\theta}{\partial u\partial v^2}+\sin\theta\frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^2-\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial^2\theta}{\partial v^2}\right.\\
&\qquad\left.-\frac{\cos\theta.\left(\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-\sin\theta\frac{\partial^2\theta}{\partial u\partial v}\right)\left(\frac{\partial^2\theta}{\partial u\partial v}-\cos\omega\cos\theta\frac{\partial\theta}{\partial v}-\frac{\partial\theta}{\partial v}\right)}{\cos\theta\frac{\partial\omega}{\partial v}+\sin\theta}\right]\frac{\frac{\partial\beta}{\partial t}}{\frac{\partial\theta}{\partial t}\frac{\partial\omega}{\partial v}}=0.
\end{aligned}$$

When this equation is differentiated with respect to u, the result reduces in consequence of (49) to

$$R\frac{\partial\beta}{\partial t}=0,$$

where R is a function which does not vanish for the above value of ω. Hence β is independent of t and the preceding equation shows that α does not involve v. Then the expression in the second parenthesis of (67) is zero*; but the latter is reducible to the form $a\cos\theta+b\sin\theta+c$, where a, b, c have the values (61), and for this to vanish for all values of θ the coefficients must be zero, from which it follows that ω must satisfy

$$(11)\qquad \frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right)=0.$$

---

* β cannot be zero, for in such case we should have from (46) and (51)

$$A=a\sin\omega,\quad C=-a\cos\omega,$$

which are the coefficients for the sphere, when a is constant.

Now equation (70) reduces to $\alpha = 0$, and from equations (69) it is seen that

$$\lambda = -\beta \sin\omega.$$

By means of (5) equation (68) can now be written

$$\cos\omega \frac{\partial C}{\partial u} + C \sin\omega \frac{\partial \omega}{\partial u} = -\beta \sin\omega \frac{\partial \omega}{\partial u},$$

from which we get by integration

$$C = -\beta + \gamma \cos\omega,$$

γ being a function of v alone. Then for A by (5) we have

$$A = -\gamma \sin\omega.$$

These values for A and C satisfy the first of equations (66), and the second reduces to

$$\gamma' = -\frac{\beta}{\sin\omega} \frac{\partial \omega}{\partial v}.$$

From (11) it follows that $\frac{1}{\sin\omega} \frac{\partial \omega}{\partial v}$ is a function of v alone, hence all the conditions will be satisfied when γ is given by

$$(71) \qquad \gamma = -\int \frac{\beta}{\sin\omega} \frac{\partial \omega}{\partial v} dv + \text{const.}$$

Hence we have the following theorem:

The necessary and sufficient condition that the lines of curvature $v = const.$ be spherical for all the transforms of an A-surface, which is not pseudospherical, is that the lines $v = const$ be plane on the original surface, and the latter be one of the infinity of surfaces with the linear element

$$ds^2 = \gamma^2 \sin^2 \omega du^2 + (\gamma \cos\omega - \beta)^2 dv^2,$$

where β is arbitrary and γ is given by (71). The linear element of a transform is

$$ds_1^2 = \beta^2 \left(\frac{\partial \theta}{\partial v}\right)^2 du^2 + \left(\beta \frac{\partial^2 \theta}{\partial v^2} + \frac{\partial \beta}{\partial v} \frac{\partial \theta}{\partial v}\right)^2 \frac{dv^2}{\left(\frac{\partial \theta}{\partial v}\right)^2}.$$

Since not more than two of these surfaces can have the lines $v = \text{const.}$ plane all the transforms of these surfaces cannot have spherical lines of curvature $v = \text{const.}$

PRINCETON UNIVERSITY.